CS 331, Fall 2025

Lecture 17 (10/27)

Today: — LP defs

— LP apps

— LP duality

Linear program definitions (Part VI, Section 4.1)

Linear function: $f(x) = c^T x = \sum_{i \in [d]} c_i x_i$

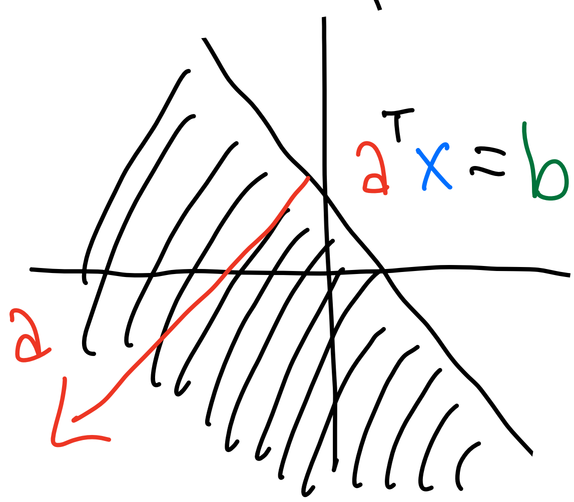
Examples $f(x) = x_1$ $\quad c = e_1$ ✓

$f(x) = 5x_2 + 7x_5$ $\quad c = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 7 \\ \vdots \end{pmatrix}$ ✓

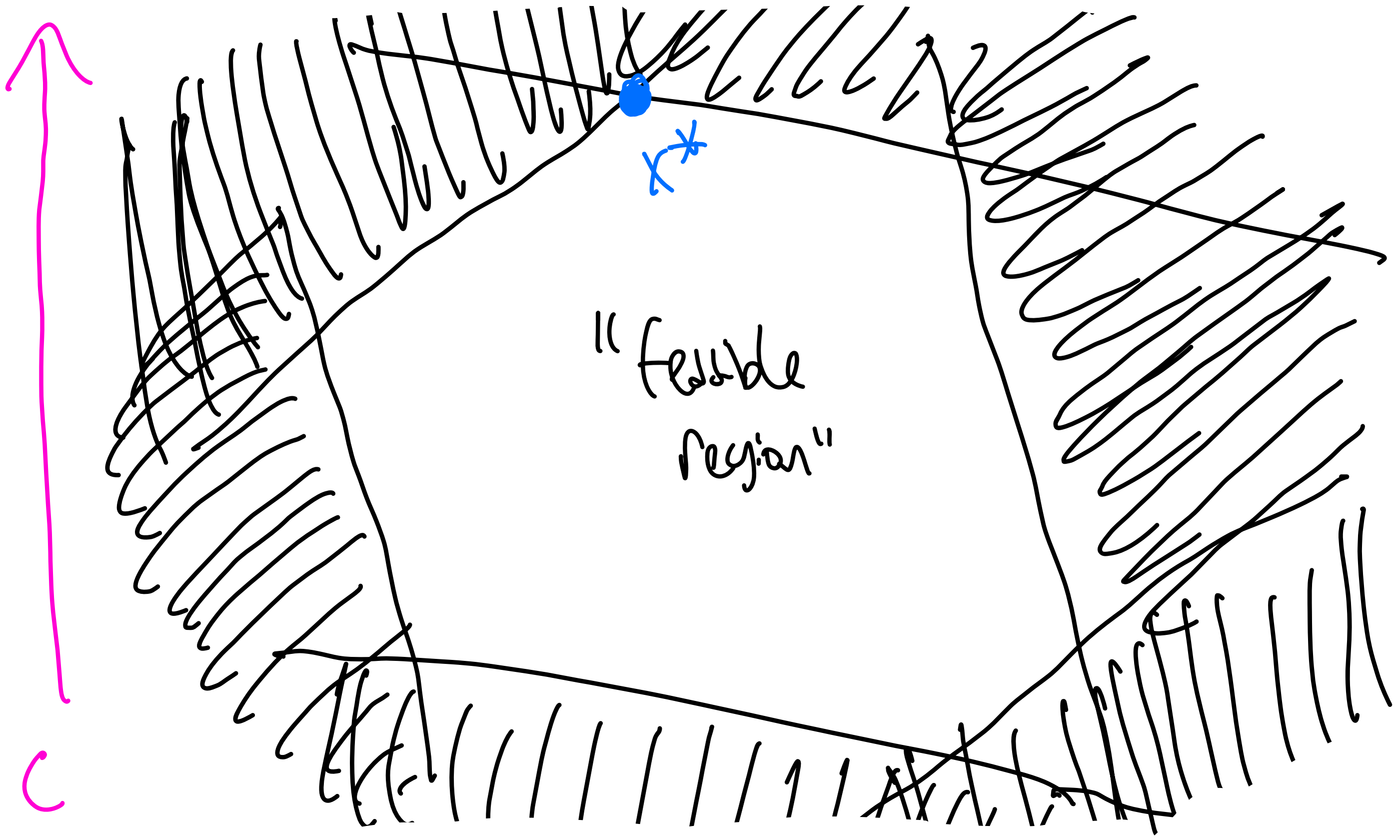
$f(x) = 3x_2^2 + 6x_2x_3$ ✗

Halfspace: $\{ x \in \mathbb{R}^d \mid a^T x \leq b \}$

Example In 2-d, $\mathbb{R}^2$, $a_1 x_1 + a_2 x_2 \leq b$



Polytope: intersection of halfspaces



LP: Optimize linear function over polytope

$$\max \; c^T x$$
$$Ax \leq b$$

$$\underset{n \times d}{A} = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix} \qquad \underset{n \times 1}{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

a.k.a. $a_i^T x \leq b_i$ for all $i \in [n]$

Much more powerful! max $c^T x$
$Ax \le b$

- min $c^T x$ OK:

  max $-c^T x$ gives same answer

- $a^T x \ge b$ OK:

  add constraint $-a^T x \le -b$

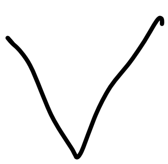
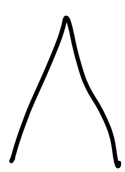
- $a^T x = b$ OK:

  add two constraints $-a^T x \le -b$
  $a^T x \le b$

- min $c^T |x|$ OK if $c \in \mathbb{R}^d_{\ge 0}$:

$z = \begin{pmatrix} x \\ y \end{pmatrix}$ add constraints: $y_i \le -x_i$ $\forall i \in [d]$
$y_i \le x_i$

new objective: max $c^T y$

Intuition:  OK to min.  not OK

LP applications (Part VI, Section 4.2)

Every LP has an equivalent dual.

Asymmetric form

$$\max_{\substack{x \in \mathbb{R}^d \\ Ax \le b}} c^T x \quad = \quad \min_{\substack{y \in \mathbb{R}^n_{\ge 0} \\ A^T y = c}} b^T y$$

(primal) (dual)

Symmetric form

$$\max_{\substack{x \in \mathbb{R}^d_{\ge 0} \\ Ax \le b}} c^T x \quad = \quad \min_{\substack{y \in \mathbb{R}^n_{\ge 0} \\ A^T y \ge c}} b^T y$$

(primal) (dual)

Exercise: rewrite symmetric as asymmetric, check dual is consistent.

Example: Resource allocation

$$n \text{ materials} \left\{ \overbrace{\begin{bmatrix} & | & \\ & A_{:j} & \\ & | & \end{bmatrix}}^{d \text{ products}} \right.$$

"recipe" for product j

$\boxed{\text{Primal LP}}$

$$\max_{\substack{x \in \mathbb{R}^d_{\geq 0} \\ Ax \leq b}} c^T x$$

b = materials available

c = market price per product unit

x = amount of each product

$$Ax = \sum_{j \in [n]} A_{:j} x_j \leq b$$

$\boxed{\text{Dual LP}}$

$$\min_{\substack{y \in \mathbb{R}^n_{\geq 0} \\ A^T y \geq c}} b^T y$$

y = offered price per material unit

$$[A^T y]_j = \sum_{i \in [n]} A_{ij} y_i \geq c_j$$

(don't sell unless beats market price)

Interpretation of duality: "packing - covering LP"

$$\max_{\substack{x \in \mathbb{R}^d_{\geq 0} \\ Ax \leq b}} c^T x \;=\; \min_{\substack{y \in \mathbb{R}^n_{\geq 0} \\ A^T y \geq c}} b^T y$$

Competitor can't beat the market if we do our best.

Example: Zero-sum games

Payoff matrix

$$A = \left.\underbrace{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}}_{\text{Bob action}}\right\}\text{Alice action}$$

If actions $= (i,j) \in [n] \times [d]$, Alice wins A_{ij}
Bob wins $-A_{ij}$

What if random strategies?

Version 0: independent

Alice picks $y \in \mathbb{R}^n_{\geq 0}$ $\quad \sum_{i \in [n]} y_i = 1$

Bob picks $x \in \mathbb{R}^d_{\geq 0}$ $\quad \sum_{j \in [d]} x_j = 1$

Expected score:

$$\sum_{\substack{i \in [n] \\ j \in [d]}} A_{ij} x_j y_i = y^T (A x) = x^T (A^T y)$$

$$(Ax)_i = \sum_{j \in [d]} A_{ij} x_j$$

Version 1: Bob-favored (can pick <u>after</u> y)

Score:

$$\max_{\substack{y \in \mathbb{R}^n_{\geq 0} \\ \sum_{i \in [n]} y_i = 1}} \underbrace{\min_{j \in [d]} [A^T y]_j}_{\text{only pick the lowest-scoring move}}$$

Version 2: Alice-favored (can pick <u>after</u> x)

Score: $\displaystyle \min_{\substack{x \in \mathbb{R}^d_{\geq 0} \\ \sum_{j \in [d]} x_j = 1}} \max_{i \in [n]} [Ax]_i$

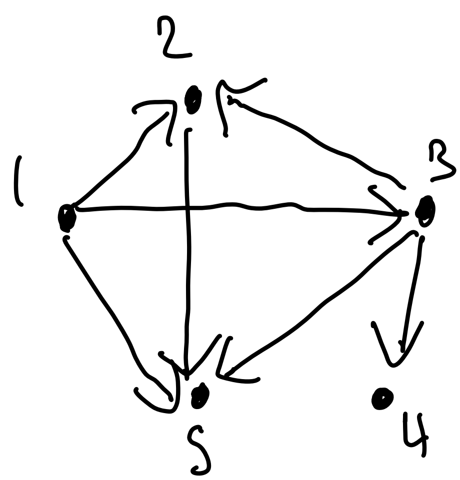
LP duality: Version 1 score = Version 2 score

(basic version of Nash equilibrium)

Example: maxflow - mincut

$A \in \mathbb{R}^{V \times E}$ edge $e = (a,b) \Rightarrow$ column $\begin{array}{cl} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} & \begin{array}{l} \\ a \\ \\ b \\ \\ \end{array} \end{array}$

$$\begin{array}{c|c}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array} &
\begin{array}{c}
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & -1
\end{pmatrix} \\
\begin{array}{ccccccc} (1,2) & (1,3) & (1,5) & (2,5) & (3,2) & (3,4) & (3,5) \end{array}
\end{array}
\end{array}$$



"incidence matrix"

Observation:

$$[Ax]_v = \sum_{(v,u)\in E} x_{(v,u)} - \sum_{(u,v)\in E} x_{(u,v)}$$

$$= \partial x(v) \quad (\text{netflow @ } v)$$

Hence maxflow LP is

$$\max_{\substack{x\in\mathbb{R}^E_{\geq 0} \\ x \leq c \\ [Ax]_U = 0}} [Ax]_s \quad \text{where } U = V \setminus \{s,t\}$$

Dual LP is

$$\min_{\substack{y\in\mathbb{R}^E_{\geq 0},\, z\in\mathbb{R}^V \\ z_s=1,\, z_t=0 \\ y_e \geq z_u - z_v \;\forall e=(u,v)\in E}} c^T y$$

Interpretation

z indicates whether vertices belong to mincut

$y_e = 1$ if crosses cut

0 else

LP duality (Part VI, Section 4.3)

Weak duality easier. Let:

$x \in \mathbb{R}^d_{\geq 0}$, $Ax \leq b$ (primal feasible)

$y \in \mathbb{R}^n_{\geq 0}$, $A^T y \geq c$ (dual feasible)

$$Ax \leq b \Rightarrow \sum_{i \in [n]} y_i (Ax)_i \leq \sum_{i \in [n]} y_i b_i$$

$$\Rightarrow y^T A x \leq b^T y$$

$$A^T y \geq c \Rightarrow \sum_{j \in [d]} x_j [A^T y]_j \geq \sum_{j \in [d]} x_j c_j$$

$$\Rightarrow y^T A x \geq c^T x$$

Hence we have $\displaystyle \max_{\substack{x \in \mathbb{R}^d_{\geq 0} \\ Ax \leq b}} c^T x \leq \min_{\substack{y \in \mathbb{R}^n_{\geq 0} \\ A^T y \geq c}} b^T y$

For strong duality, asymmetric form easier.

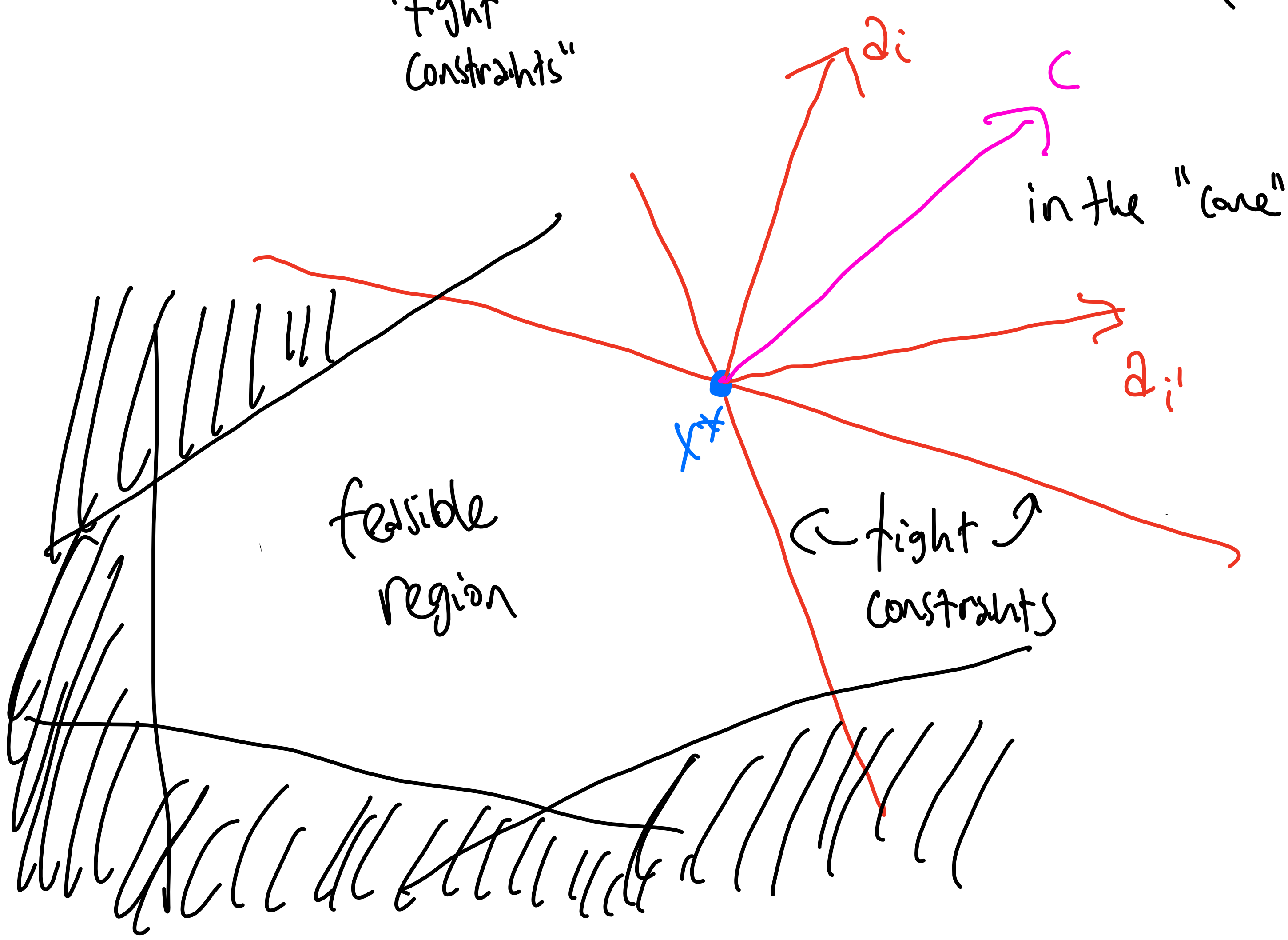
Let x^* be optimal for $\max_{Ax \le b} c^T x$

Claim: $c = \sum_{i \in I} y_i a_i \qquad y \in \mathbb{R}^n_{\geq 0}$

(Farkas' lemma)

where $I = \{ i \in [n] \mid [Ax]_i = b_i \}$

"tight constraints"



If we believe Farkas' lemma...

Let $c = \sum_{i \in [n]} y^*_i a_i$ where $y^*_i = \begin{cases} y_i & i \in I \\ 0 & i \notin I \end{cases}$

$a_i^T x^* = b_i$ for $i \in I$ (tight)

Then, $$b^T y^* = \sum_{i \in I} b_i y^*_i$$

$$= \sum_{i \in I} (Ax]_i \, y^*_i$$

$$= \sum_{i \in [n]} (Ax]_i \, y^*_i$$

$$= y^{*T} A x^* = c^T x^* \quad \blacksquare$$